Math 213: Calculus III – Exam 3

April 22, 2026

Calculators, cell phones, and formula sheets are **NOT** permitted. Show all work clearly.

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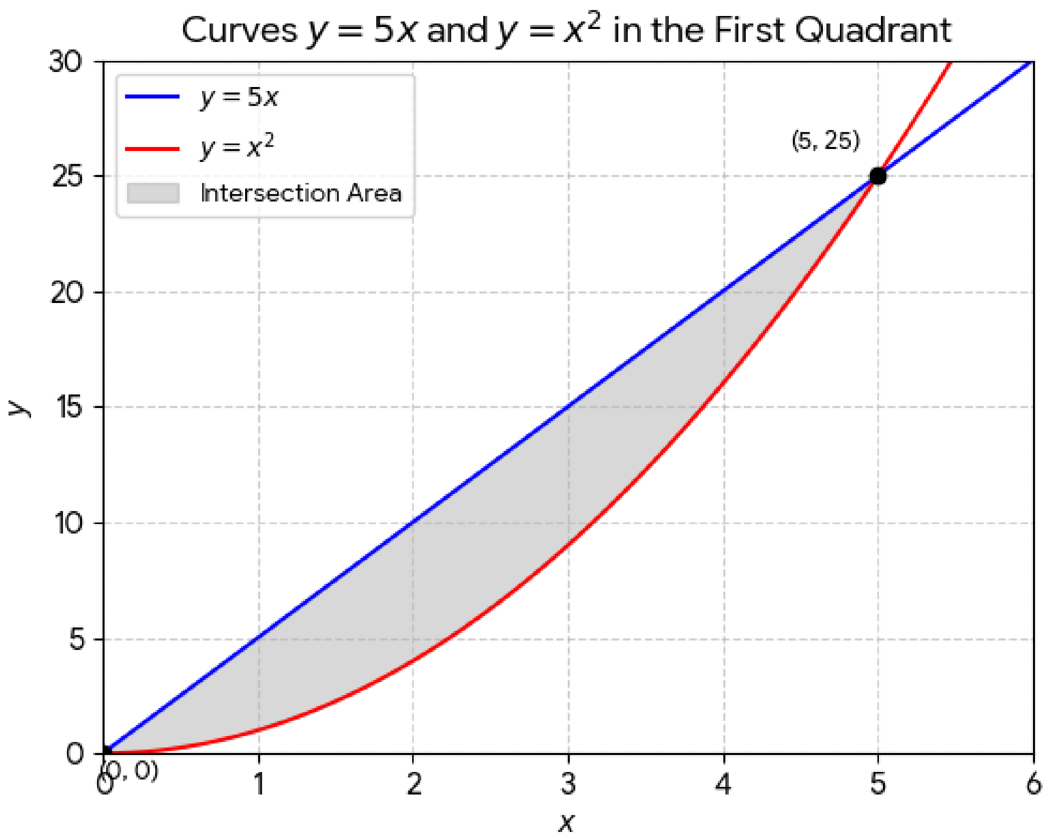
1. (10 pts): Consider the region R in the first quadrant bounded by the curves $y=x^2$ and $y = 5x$. Sketch the region R. Set up and evaluate a double integral to find the area of R.

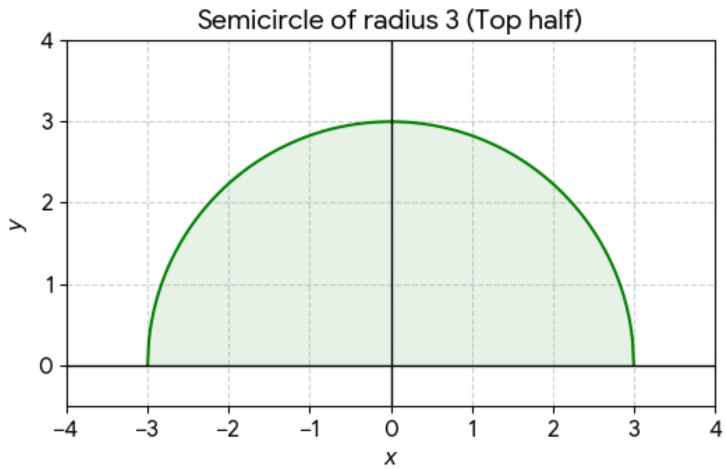


The area is $\displaystyle\int_0^5 \int_{x^2}^{5x} dydx = \int_0^5 \left(5x - x^2\right) dx = \left(\frac{5x^2}{2} - \frac{x^3}{3}\right)_0^5 = \frac{125}{6}$

Or $\displaystyle\int_0^{25} \int_{\frac{y}{5}}^{\sqrt{y}} dxdy = \int_0^{25} \left(\sqrt{y} - \frac{y}{5}\right) dy = \left(\frac{2y^{\frac{3}{2}}}{3} - \frac{y^2}{10}\right)_0^{25} = \frac{125}{6}$

2. (12 pts): Evaluate the integral by converting to polar coordinates: where R is the region in the upper half-plane ($y \geq 0$) bounded by the circle $x^2 + y^2 = 9$ and the x axis.

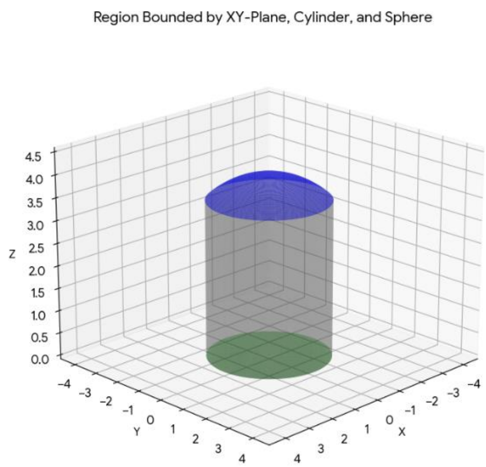
$$\iint_R (x^2 + y^2)\, dA$$



S Solution: Since $x^2 + y^2 = r^2$ and when converting to polar we have $\iint dxdy = \iint rdrd\theta$ we have

$$\iint_R (x^2 + y^2)\, dA = \int_0^{\pi} \int_0^{3} r^3 dr d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_0^3 d\theta = \frac{81\pi}{4}$$

3. (12 pts): Set up, but DO NOT EVALUATE, a triple integral using cylindrical coordinates for the mass of the solid region $\left(\int\int\int \rho dV\right)$ bounded below by the xy-plane, on the sides by the cylinder $x^2 + y^2 = 4$, and above by the sphere $x^2 + y^2 + z^2 = 16$ where the density at each point is given by $\rho(x, y, z) = e^x$

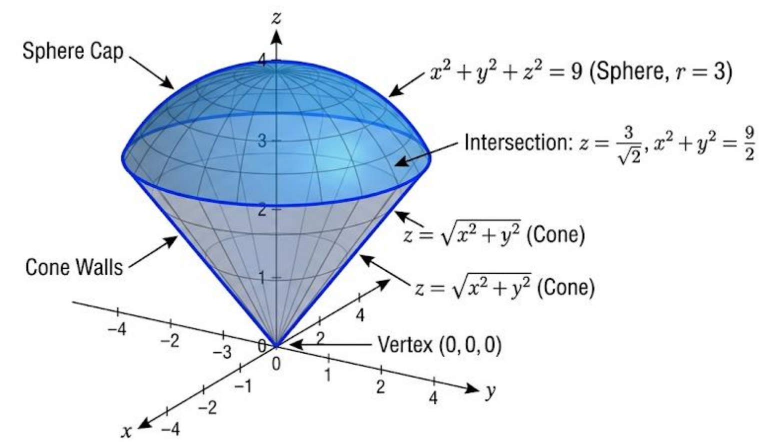


The sphere and cylinder intersect when $x^2 + y^2 = 4$ intersects $x^2 + y^2 + z^2 = 16 \Rightarrow z^2 = 12$. So we have

Mass = $\displaystyle\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{16 - x^2 - y^2}} e^x r dz dr d\theta = \int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{16 - r^2}} e^{r\cos\theta} r dz dr d\theta$

4. (12 pts) Set up, but DO NOT EVALUATE a triple integral in spherical coordinates for $\iiint z\,dV$ on the region bounded below by the cone $z^2=x^2+y^2$ and on top by the sphere $x^2+y^2+z^2=9$.

Sketch of the Region Bounded by a Cone and a Sphere in the Upper Half Plane



At the intersection: $z^2=x^2+y^2$ and $x^2+y^2+z^2=9$. Substituing, $2z^2=9 \Rightarrow z=\frac{3}{\sqrt{2}}$. The radius is $r=\frac{3}{\sqrt{2}}$.

The cone goes out at a 45 degree angle from the z-axis so we have

$$\iiint z\,dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{3}(\rho\cos\phi)\,\rho^2\sin\varphi\;d\rho\,d\phi\,d\theta$$

5. (16 pts): Show that $\vec{F}(x,y,z)=(2xy)\,\mathbf{i}+(x^2+z)\,\mathbf{j}+(y)\,\mathbf{k}$ is a conservative vector field. Then find a potential function $f(x,\ y,\ z)$ such that $\nabla f=\vec{F}$. Finally, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is any smooth path from $(0,\ 0,\ 0)$ to $(2,\ -1,\ 3)$.

The field is conservative if

$M_y=N_x;\ N_z=P_y$ and $M_z=P_x$

$2x=2x;\ 1=1;\ 0=0$

Then $f=x^2y+yz+const$ and

$$\int_C \vec{F}\cdot d\vec{r}=\int_{(0,0,0)}^{(2,-1,3)}\vec{F}\cdot d\vec{r}=f(2,-1,3)-f(0,0,0)=-4-3-0=-7$$

6. (12 pts): Evaluate the path integral $\int_C (x + \frac{1}{y+1})\, ds$ along the line segment from (0,0) to (3,2) .

The line segment can be written $x = 3t;\ y = 2t\ \Rightarrow \frac{dx}{dt} = 3;\ \frac{dy}{dt} = 2 \Rightarrow |\mathbf{v}(t)| = \sqrt{13}$.

So

$$\int_C (x + \frac{1}{y+1})\, ds = \int_0^1 \left(3t + \frac{1}{2t+1}\right)\sqrt{13}\, dt = \sqrt{13}\left(\frac{3t^2}{2} + \frac{1}{2}\ln(2t+1)\right)\Bigg|_0^1 = \sqrt{13}\left(\frac{3}{2} + \frac{1}{2}\ln 3\right)$$

=

7. (12 pts): Use Green's Theorem to find the counterclockwise circulation of the vector field $\vec{F}(x,y) = (x - y^2)\,\mathbf{i} + (x^2 - 2y)\,\mathbf{j}$ around the boundary of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. Then use Green's Theorem to find the outward flux.

For circulation, we have $M = (x - y^2)$ and $N = (x^2 - 2y)$

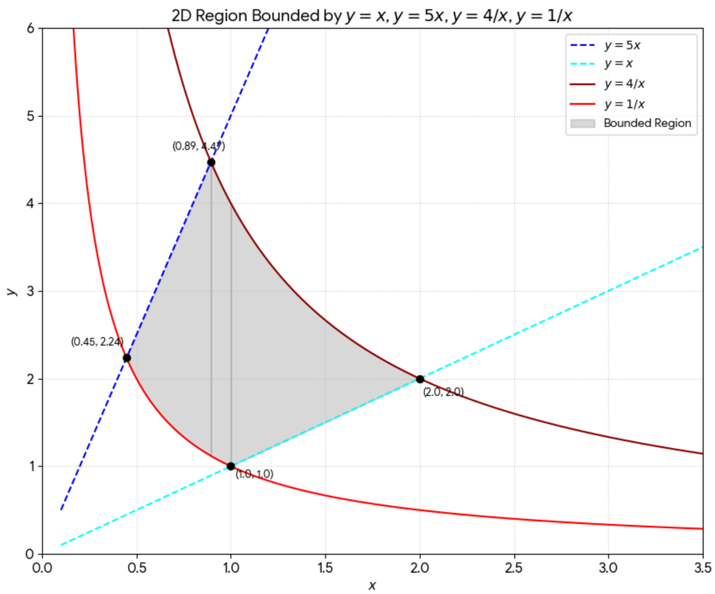
$$\oint \vec{F}(x,y)\, d\mathbf{r} = \oint M dx + N dy = \int_0^1 \int_0^1 \left(N_x - M_y\right) dxdy = \int_0^1 \int_0^1 (2x + 2y)\, dxdy =$$

$$\int_0^1 (x^2 + 2xy)\Big|_0^1 dy = \int_0^1 (1 + 2y)\, dy = (y + y^2)\Big|_0^1 = 2$$

For outward flux, we have

$$\oint \vec{F}(x,y) \cdot \mathbf{n}\, ds = \oint M dy - N dx = \int_0^1 \int_0^1 \left(M_x + N_y\right) dxdy = \int_0^1 \int_0^1 (1 - 2)\, dxdy = -1$$

8. (14 pts) Use the transformation $u = xy$ and $v = \dfrac{y}{x}$ to convert $\iint e^{xy}\,dxdy$ over the region bounded by $y = \dfrac{1}{x}$; $y = \dfrac{4}{x}$; $y = x$ and $y = 5x$ in the first quadrant into a double integral in u and v. First find x and y in terms of u and v. Then calculate the Jacobian and finally set up the integral. DO NOT EVALUATE.



Solution: Notice $uv = y^2 \Rightarrow y = \sqrt{uv}$; and $\dfrac{u}{v} = x^2 \Rightarrow x = \sqrt{\dfrac{u}{v}}$

Jacobian: $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2\sqrt{u}\sqrt{v}} & \dfrac{-\sqrt{u}}{2v\sqrt{v}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \dfrac{1}{4v} + \dfrac{1}{4v} = \dfrac{1}{2v}$

The sides become

$y = \dfrac{1}{x} \Rightarrow \sqrt{uv} = \sqrt{\dfrac{v}{u}} \Rightarrow u = 1;\quad y = \dfrac{4}{x} \Rightarrow \sqrt{uv} = 4\sqrt{\dfrac{v}{u}} \Rightarrow u = 4$

: $y = x \Rightarrow \sqrt{uv} = \sqrt{\dfrac{u}{v}} \Rightarrow v = 1$ and $y = 5x \Rightarrow \sqrt{uv} = 5\sqrt{\dfrac{u}{v}} \Rightarrow v = 5$

Finally $\iint e^{xy}\,dxdy$ becomes $\displaystyle\int_1^4\int_1^5 \frac{1}{2v}e^{u}\,dvdu$ or $\displaystyle\int_1^5\int_1^4 \frac{1}{2v}e^{u}\,dudv$